Some early ideas about types

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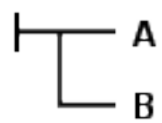
Gottlob Frege, ***Begriffsschrift***, 1879

Propositional Logic
Quantifiers
2-dimentional notation



(lost out vs Peano’s linear notation)

Propositional functions:
a proposition viewed as a function applied to arguments.

Objects:
things that propositional functions can be applied to.

Examples:

The sky is blue. - a proposition, can be true or false

Viewing it as a *propositional function*: What is the argument?

sky has_color (blue)   -- blue is the argument

Φ(A) == sky has color A

(sky) has_color blue   -- sky is the argument

Φ(A) == A has color blue

(sky) has_color (blue) -- both sky and blue are argument
(2-arg prop function)

Another example

x < 3, a propositional function of one argument, x

Propositional functions are *intentional:*

The application of a propositional function produces a proposition (*statement) which can be true or false*.

Propositional functions may have several arguments

A has_color B

Propositional functions as variables in propositions (almost).

Φ(A,B)

Here Φ represents an *indeterminate propositional function* of two arguments, i.e. a variable ranging over propositional functions, hinting at propositional functions as arguments.

Can propositional functions be "objects" that other propositional functions apply to? e.g. $\Psi(\Phi)$

No. Propositional functions take "objects" as arguments, and propositional functions are not considered ”objects".

Later, in **Foundations of Arithmetic** (*1893*), Frege allowed the “*course of values*” (i.e. extension, or graph) of a propositional function to be treated as an object, and this opened the door for Russell’s paradox: A propositional function might be applied to its own extension as an object.

Notation for course of values: ‘$\varepsilon\Phi(\varepsilon)$

==> ^xΦ(x) ==> $\wedge$xΦ(x) (= $\{x \mid \Phi(x)\}$) (Russell)

=?=> $\wedge$x.e(x) (Church)

Foreshadowing of types in Frege’s treatment of propositional functions:

* Multi-argument types apply to pairs, triples, etc, suggesting product types

* First-order pfs apply to “individuals”, 2nd order pfs apply to first-order pfs, etc.

Russell's paradox, discovered June, 1901

$S = \{ x \mid x \notin x \}$ $\qquad S \in S \quad <=> \quad S \notin S$

$\Psi(\Phi)$ iff not $\Phi(|\Phi|)$. $\Psi(|\Psi|) \;<=>\;$ not $\Psi(|\Psi|)$

where $|\Psi|$ is the extension of Ψ

* Correspondence between Russell and Frege, June 1992.

Bertrand Russell, **Principles of Mathematics**,1903, Appendix B

Here Russell first proposes types as a potential solution to the paradoxes.

Preface:

"In the case of classes, I must confess, I have failed to perceive any concept fulfilling the conditions requisite for the notion of *class*. And the contradiction discussed in Chapter x. proves that something is amiss, but what it is I have hitherto failed to discover."

Appendix B: toward a *theory of types*

"Every propositional function Φ(x) — so it is contended — has, in addition to its range of truth, a range of significance, i.e. a range within which x must lie if Φ(x) is to be a proposition at all, whether true or false. This is the first point in the theory of types; the second point is that ranges of significance form types, i.e. if x belongs to the range of significance of Φ(x), then there is a class of objects, the type of x, all of which must also belong to the range of significance of Φ(x), however Φ may be varied; and the range of significance is always either a single type or a sum of several whole types. The second point is less precise than the first, and the case of numbers introduces difficulties; but in what follows its importance and meaning will, I hope, become plainer."

Thus a type is viewed as the ***range of significance*** of a propositional function (not the range of *truth* of a propositional function).

Products: propositional functions over two or three arguments
=> The range consists of pairs, triples, respectively.

Russell developed types further in

"*Mathematical logic as based on the theory of types*", 1908.

Claims *impredicativity* is at the root of all the paradoxes
— a kind of self-referential definition.

Russell's **Vicious Circle Principle:**

> "Whatever involves all of a collection must not be one of the collection."

He avoids impredicativity by defining orders of propositions, where a proposition is of *order* n+1 if it contains a universal quantifier over variables ranging over things of order n

This was called *ramification* (or *ramified types*) when used in conjunction with a *simple type theory* in **Principia Mathematica** (1910-1913).

The simple type theory was not defined, but involved

* products — types consisting of tuples

* higher “kinds”: 1st order pfs, 2nd order pfs, etc.

Types are not defined, nor is there a notation to express them. Only a of “being of the same type” is defined (incompletely).

1920s: Ramsey and Hilbert and Ackermann

* The simple theory of types suffices to avoid the paradoxes.

* Ramsey (1926) gave an explicit definition of simple types

  - 0 is a simple type (ST)
  - t1, …, tn are ST => (t1,…tn) is a ST

These describe the argument types of propositional functions.

E.g. (0, (0,0))
= the type of a pf that takes an individual
and a pf with two individual arguments

Meanwhile, in **Set Theory:**

Zermelo 1908 + Fraenkel 1921 ==> ZF axiomatization

Russell's paradox is killed by

* axiom of *comprehension*

  $\{x \in A \mid \Phi(x)\}$, not $\{x \mid \Phi(x)\}$

* for extra measure, axiom of *foundation*

  $\in$ is well-founded ==> not $s \in s$

NBG (von Neumann, Bernays, Gödel)
axiomatic set theory with sets and classes

1934: Curry: *Functionality* in Combinatory Logic
Curry's type theory for Combinatory Logic

* primitive F combinator for constructing function types

$$\mathrm{Fabf} \quad \simeq \quad f : a \rightarrow b$$

Semantics: $(\forall x)(x \in a \Longrightarrow f(x) \in b)$

If x is a type, y a term, xy asserts that y has type x, $(y \in x)$

Axiom F: $(\forall x,y,z)(\mathrm{F}xyz \Longrightarrow (\forall u)(xu \Longrightarrow y(zu))$

Curry's *Functionality* for CL

* types used as predicates (propositional functions) that can apply to "value" expressions. Distinction between variables representing types and variables representing values is implicit.

* types are things that can be asserted (proved) to apply to terms

* combinators (hence terms) can have many types (polymorphism!) Typing Axioms assign *type schemes* with *universally quantified* type variables (polymorphic types!) to basic combinators:

Axioms for typing primitive combinators, e.g.

[FK] $\forall(x, y)\ Fy(Fxy)K$ K : $\forall(x, y)\ y \rightarrow (x \rightarrow y)$

Note: this paper also defines a precursor of the Y combinator, used to show that Russell's paradox is avoided, because of the types.

1940: Church: The Simple Theory of Types

simple theory of types combined with the lambda calculus

still treated as a logical language for reasoning

two primitive types, ι (individuals), o (propositions, or Bool) and function types, e.g.

$= : \iota \to \iota \to o$ (a binary relation on individuals)

$\neg : o \to o$

$\forall : (\iota \to o) \to o$ (universal quantification)

The End

1969: Curry: Modified basic *functionality* in Combinatory Logic

1. types for combinators are called *functional characters*

2. type expressions are called **F-obs**: Fab = a $\rightarrow$ b
   ranged over by metavariable χ

3. primitive types are called F-simples (e.g. N = Nat)

4. "indeterminate F-simples" = parameters = type variables

5. "F-schemes" are type expressions possibly containing type
   variables (parameters)

6. typing judgement of form $\vdash \chi X$, where χ is an F-ob and
   X is a combinator term

   χ is a "functional character of X", i.e. a type (scheme) for X

   "X is stratified" = "X is well-typed" = there exists χ s.t. |- χ X

Rules: There is a single rule, namely

RULE F:

$\vdash$ (F $\chi\ \epsilon$) X & $\vdash \chi$ Y $\Longrightarrow$ $\vdash \epsilon$ (XY)

This is the usual -> elimination rule for function application.

Typing rules for primitive (constant) combinators I, K, and S given by the axioms:

[FI] $\vdash$ F $\alpha\ \alpha$ I   i.e. I : $\alpha \to \alpha$

[FK] $\vdash$ F α (F $\beta\ \alpha$) K   i.e. K : $\alpha \to (\beta \to \alpha)$

[FS] $\vdash$ F (F α (F $\beta\ \gamma$)) (F (F $\alpha\ \beta$) (F $\alpha\ \gamma$)) S

i.e. S : $(\alpha \to (\beta \to \gamma)) \to (\alpha \to \beta) \to \alpha \to \gamma$